ENGR352 Problem Set 02

Transfer function of an estimator

1. Using Eq. (1.1.4-27) from the text, find the correct value of $\tilde{r}_{ss}$ (the result given in the text is incorrect).

2. Obtain the correct $\tilde{G}$ in (1.1.4-29).

3. Obtain the correct $\tilde{H}_1(z)$ in (1.1.4-30).

4. With the correct $\tilde{H}_1(z)$ obtain the steady state position error $\tilde{x}_{1_{ss}}$ for the input being a constant (step function) acceleration $u(k) = u_0 = 10\text{m/s}^2$ using the z-transform final value theorem.

5. Is the result from 4 consistent with the result from 1? How could one obtain directly the result for 4 from the result of 1?

ENGR352 Problem Set 03

1. **Matrix inversion lemma.** Verify (1.3.3-11) by multiplication, which has to yield the identity matrix.

2. **Quadratic forms.** Given the matrices

$$A_1 = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

evaluate

$$q_i = x' A_i x \qquad i = 1, 2, 3$$

where

$$x = [x_1 \;\; x_2]'$$

and explain the result.

ENGR352 Problem Set 04

1. **Controllability and Observability of Linear Dynamic Systems.** Given the dynamic system (1.3.8-1)–(1.3.8-2) with

$$F(k) = F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

$$G(k) = G = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

$$H(k) = H_1 = [1 \;\; 0]$$

$$H(k) = H_2 = [0 \;\; 1]$$

(i) Is the system $\{F, G, H_1\}$ completely controllable and completely observable?

(ii) Is the system $\{F, G, H_2\}$ completely controllable and completely observable?

(iii) If the state components are x_1=position and x_2=velocity, explain in words the observability results.

2. **Normalization of Probability Density Functions.** Find the normalization constants for the following pdfs:

(i) Exponential

$$p(\tau) = c_1 e^{-\lambda\tau} \qquad \tau \in [0, \infty)$$

(ii) Laplace

$$p(\tau) = c_2 e^{-\lambda|\tau|} \qquad \tau \in (-\infty, \infty)$$

(iii) Rayleigh

$$p(\tau) = c_3 \tau \, e^{-\dfrac{\tau^2}{2a^2}} \qquad \tau \in [0, \infty)$$

ENGR352 Problem Set 04

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(i) Is the system $\{F, G, H_1\}$ completely controllable and completely observable?

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ENGR352 Problem Set 05

1. **Accuracy of a poll**. Given the binary i.i.d. (independent, identically distributed) random variables x_i, with

$$P\{x_i = 1\} = p = 1 - P\{x_i = 0\}$$

(i) Find the mean and s.d. (standard deviation) of

$$z = \frac{1}{n} \sum_{i=1}^{n} x_i$$

(ii) Evaluate the double of the s.d. for $n = 1200$ and $p = 0.5$ and express it as a percentage.

2. Problem 1-2 from text.

3. **Average power dissipated in a resistor subjected to a random voltage**. A DC voltage V is a random variable with a symmetric triangular pdf in the interval $[V_0 - a,\, V_0 + a]$ (in volts) and zero outside it.

(i) Write the expression of this pdf.

(ii) Find its mean and variance.

(iii) Find the expected value of the power dissipated when it is applied to a resistor $R = 1\Omega$.

(iv) What is the extra percentage power in (iii) for $a/V_0 = 0.2$ vs. the nonrandom case where $V = V_0$.

ENGR352 Problem Set 06

1. **Reliability and Life of a Mechanism.** Given that the time τ until a mechansim breaks down (its "life") is exponentially distributed

$$p(\tau) = e^{-\tau} \qquad \tau \in [0, \infty)$$

(i) Find its expected life (use integration by parts)

(ii) Find the pdf of its "remaining life" given that it did not break down up to $\tau = \tau_0 = 1$ (the remaining life is $\tau_1 = \tau - \tau_0$).

(iii) Find the expected value of the remaining life τ_1 according to (ii).

(iv) Comment of the suitability of the exponential distribution for this problem.

2. Repeat Problem 1 for a Rayleigh distributed life

$$p(\tau) = \frac{\tau}{a^2}\, e^{-\dfrac{\tau^2}{2a^2}} \qquad \tau \in [0, \infty)$$

where $a = \sqrt{2/\pi}$.

For item (iii) it is recommended to use integration by parts as well as numerical integration to verify the result.

ENGR352 Problem Set 07

1. **Accuracy of Estimation.** Given the jointly Gaussian scalar random variables

$$x \sim \mathcal{N}(\mu_x, \sigma_x^2) \qquad y \sim \mathcal{N}(\mu_y, \sigma_y^2)$$

with

$$\mathrm{cov}(x, y) = \rho\, \sigma_x \sigma_y$$

(i) Find the values of their correlation coefficient ρ such that the conditional variance of x given y is half of its unconditional variance, i.e.,

$$\mathrm{var}(x|y) = 0.5\, \mathrm{var}(x)$$

(ii) What is the smallest achievable $\mathrm{var}(x|y)$? What values of ρ yield this?

(iii) Give the relationship between x and y when the requirement of (ii) is satisfied.

2. **Autocorrelated Random Process.** Given a dynamic system of the form (1.4.19-11) to be used as a model for an autocorrelated stationary process (a voltage) with the following properties:

(i) The autocorrelation coefficient of this process between for two points in time 10s apart is 0.1. Using this, find the parameter a of the model. Indicate its units.

(ii) Its instantaneous m.s. value should be $5\mathrm{V}^2$. Find the power spectral density (psd) S_0 of the white noise driving this system. Indicate its units.

(iii) What are the units of the noise $n(t)$ in (1.4.19-11)? Show that the units of its psd obtained above are the units of $n(t)$ squared and divided by the frequency units (Hz).

ENGR352 Problem Set 08

1. **A Markov Chain Driven System.** A system exhibits two behavior modes. The transition probabilities between its two modes are given by the elements of the transition probability matrix (see Eq. (1.4.22-14) in the text)

$$\Pi = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

(i) Find the steady state of its probability vector (defined in Eq. (1.4.22-10) in the text).

(ii) What is the ratio of the time this system spends in state 1 vs. in state 2?

2. **Pipelines and the Central Limit Theorem.** A pipeline is to be assembled from N sections whose lengths x_i are i.i.d. random variables with uniform distribution

$$x_i \sim \mathcal{U}(x_0 - \epsilon,\, x_0 + \epsilon)$$

(i) Find the mean and variance of x_i.

(ii) Using the Central Limit Theorem find the probability that, for $N = 100$ pipes with nominal length $x_0 = 10$m and tolerance $\epsilon = 1$cm, the total length L differs by more than 9.5cm from the nominal one.

ENGR352 Problem Set 09

1. **Probability Concentration Regions.** A variable has a chi-square distribution with $n = 400$ degrees of freedom.

(i) Find its one-sided 95% probability region (cutting its upper 5% tail).

(ii) Find its two-sided 95% probability region (cutting its upper and lower 2.5% tails).

(iii) Find the two-sided 95% region for

$$y_n = x_n/(n/4)$$

and its relative size from $E\{y_n\}$.

(iv) Repeat (iii) for $n = 800$.

(v) Find the ratio of the two relative interval lengths in (iii) and (iv).

(vi) Could you have predicted this ratio based on the two values of n?

ENGR352 Problem Set 10

1. **From prior to posterior pdf.** Given the observation $z = 10$ of the variable x, with pdf

$$p(z|x) = \mathcal{N}(z; x, \sigma^2)$$

with $\sigma = 10$ and the prior of x as

$$p(x) = \mathcal{N}(x; x_0, \sigma_0^2)$$

with $x_0 = 5$ and $\sigma_0 = 20$.

(i) Find the estimate $\hat{x}^{\text{MAP}}$.

(ii) Find the variance of the posterior pdf $p(x|z)$.

(iii) Write the expresssion of posterior pdf $p(x|z)$ with the above numbers.

2. **Estimation of the parameter of a pdf.** Given a Rayleigh distributed observation

$$p(z) = \frac{z}{a^2}\, e^{-\dfrac{z^2}{2a^2}}$$

with unknown parameter a.

(i) Find the MLE of a based on a single observation z.

(ii) Find the MLE of a based on the i.i.d. observations z_i, $i = 1, \ldots, n$, with the above pdf.

ENGR352 Problem Set 11

Problem 2-4 from the text.

ENGR352 Problem Set 12

1. Problem 2-12 from the text.
2. Problem 2-14 from the text.

ENGR352 Problem Set 13

1. **Information Fusion and evaluation of its benefit.** Given the covariance matrix characterizing the quality of the initial (prior) information (from source 1) about a vector x as

$$P_{xx} = E[(x - \bar{x})(x - \bar{x})'] = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

where $\bar{x}$ is the initial estimate (initial mean).

A sensor (source 2) is used to make the following scalar observation on x

$$z = Hx + w$$

where

$$H = [1\ 1]$$

and the noise w has moments

$$E\{w\} = \bar{w} = 0 \qquad\qquad E\{w^2\} = \sigma_w^2 = 1$$

and is independent of the error in the initial estimate $\bar{x}$.

(i) Evaluate the mean $\bar{z}$ of z based on the initial mean of x and the properties of w.

(ii) Evaluate the covariance matrices

$$P_{zz} = E[(z - \bar{z})(z - \bar{z})'] = E[(z - \bar{z})^2]$$

and

$$P_{xz} = E[(x - \bar{x})(z - \bar{z})']$$

Note: these results should be independent of the initial estimate of x.

(iii) Using the fundamental equations of linear estimation (Sec. 3.3), evaluate the covariance (MSE matrix) $P_{xx|z}$ of the "fused estimate" resulting from combining the information from the two sources.

(iv) Compare the uncertainty volumes (which are proportional to the square roots of the determinants of the covariance ematrices) before fusion (i.e., based on the prior) and after fusion (based on the posterior).

ENGR352 Computer Project 14

Localization based on range measurements (simplified GPS problem).

There are 4 satellites whose locations w.r.t. an ECEF (Earth Centered Earth Fixed) coordinate system are (in km)

$$x_{S_1} = [0 \quad 0 \quad 15,000]'$$

$$x_{S_2} = [15,000 \quad 0 \quad 8,000]'$$

$$x_{S_3} = [0 \quad 15,000 \quad 8,000]'$$

$$x_{S_4} = [10,000 \quad 10,000 \quad 8,000]'$$

Your unknown location x has the true value

$$x_0 = [\xi_0 \quad \eta_0 \quad \zeta_0]' = [10 \quad 10 \quad 6,380]'$$

A range (distance) measurement is made from from each satellite to your location

$$z_i = d(x_0, x_{S_i}) + w_i \qquad i = 1, \ldots, 4$$

where $d(x_0, x_{S_i})$ is the true distance, with i.i.d. measurement errors

$$w_i \sim \mathcal{N}(0,\, \sigma_w^2)$$

where $\sigma_w = 0.02$km.

(i) Write the ILS equations to estimate your location x, with true value x_0.

(ii) Using a random number generator, generate a set of noisy measurements as specified above. Using the ILS method, find the LS estimate $\hat{x}$ of x based on these measurements; use as initial estimate the vector $[0 \quad 0 \quad 6,370]$. Repeat this $N = 100$ times (with independent random variables from run to run) and find the mean square value of the estimation errors and RMS value for each coordinate separately.

(iii) Evaluate the (theoretical) MSE for each component of the LS estimates for this problem. Compare with the results from (ii). Are they statistically compatible?

ENGR352 Problem Set 15

1. **Information matrix form of Fusion.** Use (3.4.2-6) to verify the result you got in Problem Set 13, item 1(iii). Hint: $P(k)^{-1}$ plays the role of the prior information, while $P(k+1)^{-1}$ plays the role of the posterior information.

ENGR352 Problem Set 16

1. Problem 3-9 from the text.
2. Problem 3-10 from the text.

ENGR352 Problem Set 17

1. Problem 3-11 from the text.
2. Problem 3-12 from the text.

ENGR352 Problem Set 18

1. Problem 4-3 from the text.
2. Problem 4-4 from the text.

ENGR352 Problem Set 19

1. Problem 4-8 from the text.

ENGR352 Problem Set 21

1. **Joseph form of the state MSE matrix recursion.**

(i) Prove (5.2.3-18) from the text. Hint: express $\tilde{x}(k+1|k+1)$ in terms of the previous prediction error $\tilde{x}(k+1|k)$ using an arbitrary filter gain.

(ii) Show the recursion of $\tilde{x}(k+1|k+1)$ in terms of the previous estimation error $\tilde{x}(k|k)$ for an arbitrary filter gain.

(iii) Find the recursion of the MSE matrix $P(k+1|k+1) \triangleq E[\tilde{x}(k+1|k+1)\tilde{x}(k+1|k+1)']$ in terms of the previous MSE matrix $P(k|k)$ for an arbitrary filter gain. (Note: P is not a covariance anymore because the estimate is not optimal — not the conditional mean anymore, due to the arbitrary gain).

(iv) What is the stability condition for (iii)?

ENGR352 Problem Set 22

1. Problem 5-1 from the text.
2. Computer Application 5-2 from the text.

ENGR352 Problem Set 23

Problem 5-14 from the text.

Problem 5-15 from the text.

ENGR352 Problem Set 24

Problem 5-16 from the text.

Problem 5-17 from the text.

ENGR352 Problem Set 26

Problem 6-1 from the text.

ENGR352 Problem Set 27

1. Problem 6-8 from the text.
2. Problem 6-9 from the text.

ENGR352 Problem Set 28

1. Problem 10-1 from the text.
2. Problem 10-2 from the text.

ENGR352 Problem Set 29

1. Problem 10-7 from the text.
2. Problem 10-8 from the text.

ENGR352 Problem Set 30

1. Problem 10-10 from the text.
2. Problem 10-11 from the text.

ENGR352 Problem Set 32

Problem 11-2 from the text.

ENGR352 Problem Set 33

Problem 11-6 from the text.

Problem 11-7 from the text.

ENGR352 Problem Set 35

Problem 11-9 from the text.

ENGR352 Computer Project 38

Localization based on range measurements with clock bias (GPS problem).

There are 6 satellites whose locations w.r.t. an ECEF (Earth Centered Earth Fixed) coordinate system are (in km)

$$x_{S_1} = [0 \quad 0 \quad 15,000]'$$

$$x_{S_2} = [15,000 \quad 0 \quad 8,000]'$$

$$x_{S_3} = [0 \quad 15,000 \quad 8,000]'$$

$$x_{S_4} = [10,000 \quad 10,000 \quad 8,000]'$$

$$x_{S_5} = [-10,000 \quad -10,000 \quad 8,000]'$$

$$x_{S_6} = [-5,000 \quad -10,000 \quad 10,000]'$$

Your unknown location x has the true value

$$x_0 = [\xi_0 \quad \eta_0 \quad \zeta_0]' = [10 \quad 10 \quad 6,380]'$$

A noisy pseudorange measurement (pseudodistance — based on propagation time measured with an imperfect clock) is made from each satellite to your location x

$$z_i = h(x, b, x_{S_i}) + w_i = d(x, x_{S_i}) + b + w_i \qquad i = 1, \ldots, 6$$

where

$$d(x, x_{S_i}) = \|x - x_{S_i}\|$$

is the Cartesian distance from x to x_{S_i}, with i.i.d. measurement errors (noises)

$$w_i \sim \mathcal{N}(0, \sigma_w^2)$$

where $\sigma_w = 0.01$km and an unknown constant timing bias b (expressed as a distance error, i.e, the local clock error is already multiplied by the propagation speed).

The vector to be estimated is

$$\mathbf{x} = [x' \ b]'$$

The true (unknown) value of the bias is $b_0 = 0.03$km.

Note: The expression of the measurement z_i is written in terms of the unknowns x and b, which are the variables that will appear in the estimation algorithm. The true values, subscripted by 0, are to be used *only* for the error calculation in the algorithm evaluation.

(i) Write the ILS equations to estimate $\mathbf{x}$ (your location and timing bias).

(ii) Using a random number generator, generate a set of noisy measurements as specified above. Using the ILS method, find the LS estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ based on these measurements; use as initial location estimate the vector $[0 \quad 0 \quad 6,370]$ and for the bias use zero as initial estimate. Repeat this $N = 100$ times (with independent random variables from run to run) and find the mean square value of the estimation errors and RMS value for each component separately.

(iii) Evaluate the (theoretical) MSE of the LS estimates for this problem. Compare with the results from (ii). Are they statistically compatible?

(iv) Evaluate the (theoretical) RMSE of the horizontal error, the horizontal and vertical DOP.

(v) Repeat (ii)–(iv) assuming only satellites 2,3,4,5 are available. Why is the VDOP so different?

ENGR352 Final Problem

Multisource Information Fusion

Assume you are receiving the output of a Kalman Filter based on a DWNA (direct discrete time white noise acceleration) motion model with position observations in the presence of additive zero mean white measurement noise with $\sigma_w = 10$m. Assume the process noise to be independent of the measurement noise, zero mean, white and with $\sigma_v = 0.25\text{m/s}^2$. The sampling interval is $T = 2$s.

An external noisy measurement

$$y = Cx + n$$

is obtained from an independent source, where

$$C = [c_1 \;\; c_2]$$

and

$$n \sim \mathcal{N}(0, \sigma_e^2)$$

where $\sigma_e = 5$m.

(i) Evaluate the steady state (s.s.) covariance $\bar{P}$ from this filter.

(ii) Develop the appropriate information fusion equations between the (s.s.) output of the above filter $(\bar{x}, \bar{P})$ at a certain time and the external measurement y (assumed to be made at the same time) using the fundamental equations of linear estimation (Sec. 3.2.1). Give the expressions of the fused estimate $\hat{x}$ and its asociated covariance matrix $\hat{P}$.

(iii) Evaluate the covariance of the fused state estimate for case 1: $c_1 = c_2 = 1$.

(iv) Evaluate the covariance of the fused state estimate for case 2: $c_1 = -c_2 = 1$.

(v) Which case yields smaller final errors? Can you provide an explanation of the result?