ENGR352 Problem Set 02 SOLUTION

1.

$$\tilde{r}_{ss} = 0.025\text{m}$$

2.

$$\tilde{G} = \begin{bmatrix} 0.00125 \\ 0.075 \end{bmatrix}$$

3.

$$\tilde{H}_1(z) = \frac{0.00125(z+1)}{z^2 - 0.75z + 0.25}$$

4.

$$U(z) = \frac{10z}{z-1}$$

$$\tilde{X}_1(z) = U(z)\tilde{H}_1(z)$$

$$\tilde{x}_{1_{ss}} = \lim_{z\to 1}(z-1)\tilde{X}_1(z) = 0.05\text{m}$$

5. The result from 4 is consistent with the result from 1. Using the linearity property, it could be obtained directly from the result of 1 by multiplying the latter by 2, because the input in 4 is twice the input in 1.

ENGR352 Problem Set 03 SOLUTION

1. Multiplying the two sides of (1.3.3-11) one obtains

$$(P^{-1} + H'R^{-1}H)[P - PH'(HPH' + R)^{-1}HP] = I + T_2 + T_3 + T_4$$

where

$$T_2 = -H'(HPH' + R)^{-1}HP$$

$$T_3 = H'R^{-1}HP$$

$$T_4 = -H'R^{-1}HPH'(HPH' + R)^{-1}HP$$

Then

$$\begin{aligned} T_2 + T_3 &= -H'(HPH' + R)^{-1}HP + H'R^{-1}HP \\ &= -H'[(HPH' + R)^{-1} - R^{-1}]HP \\ &= -H'[I - R^{-1}(HPH' + R)](HPH' + R)^{-1}HP \\ &= -H'[I - R^{-1}HPH' - I](HPH' + R)^{-1}HP \\ &= H'R^{-1}HPH'(HPH' + R)^{-1}HP \\ &= -T_4 \end{aligned}$$

i.e., one has only the identity matrix left from the multiplication.

2.

$$q_i = x'A_ix = x_1^2 + 6x_1x_2 + 2x_2^2 \qquad i = 1, 2, 3$$

because

$$A_3 = (A_1 + A_1')/2 = (A_2 + A_2')/2$$

In general

$$x'Ax = x'[(A + A')/2 + (A - A')/2]x = x'[(A + A')/2]x$$

because $A - A'$ has zero diagonal terms and is antisymmetric. Consequently,

$$x'[(A - A')/2]x = 0$$

and this explains the result.

ENGR352 Problem Set 04 SOLUTION

1.

(i)

$$\mathcal{Q}_C = \begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix} \qquad \text{rank}(\mathcal{Q}_C) = 2$$

$$\mathcal{Q}_O = \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix} \qquad \text{rank}(\mathcal{Q}_O) = 2$$

i.e., completely controllable and completely observable.

(ii)

$$\mathcal{Q}_C = \begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix} \qquad \text{rank}(\mathcal{Q}_C) = 2$$

$$\mathcal{Q}_O = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{rank}(\mathcal{Q}_O) = 1 < 2$$

i.e., completely controllable but NOT completely observable.

(iii) The complete observability in case 1 is a consequence of the fact that from position-only observations one can infer the velocity (by differencing) but from velocity-only observations one cannot infer the position because the intial position (the constant in the integration from velocity to position) is not known.

2. The integral of each pdf over its domain has to be unity.

(i)

$$c_1 = \lambda$$

(ii)

$$c_2 = \lambda/2$$

(iii)

$$c_3 = 1/a^2$$

ENGR352 Problem Set 05 SOLUTION

1.

(i)

$$E\{x_i\} = 1 \cdot p + 0 \cdot (1-p) = p$$

$$E\{x_i^2\} = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

The s.d. of x_i is

$$\sigma_{x_i} = \sqrt{E\{x_i^2\} - [E\{x_i\}]^2} = \sqrt{p - p^2} = \sqrt{p(1-p)}$$

$$E\{z\} = \frac{1}{n}\sum_{i=1}^{n} E\{x_i\} = p$$

$$\sigma_z = \sqrt{\frac{1}{n^2}[n\sigma_{x_i}^2]} = \sqrt{\frac{p(1-p)}{n}}$$

(ii)

$$2\sigma_z = 2\sqrt{\frac{0.5(1-0.5)}{1200}} \approx 3\%$$

2. Let

$$\bar{p} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Then, relying on $\hat{p}$, one has

$$\sigma_{\bar{p}} = \sqrt{\sigma^2/N} = \sqrt{\frac{\hat{p}(1-\hat{p})}{N}} \approx \sqrt{\frac{\hat{p}}{N}}$$

Confirmation can be taken as

$$\left|\frac{\bar{p} - \hat{p}}{\hat{p}}\right| \leq 0.1$$

To obtain this with 95% probability one needs

$$\frac{2\sqrt{\hat{p}/N}}{\hat{p}} \leq 0.1$$

which yields

$$N \geq 4 \cdot 10^6$$

3.

(i)

$$p(V) = \begin{cases} \frac{V - V_0 + a}{a^2} & V_0 - a < V < V_0 \\ -\frac{V - V_0 - a}{a^2} & V_0 < V < V_0 + a \\ 0 & \text{otherwise} \end{cases}$$

(ii)

Because of symmetry, $E\{V\} = V_0$. The variance is, with $x = V - V_0$,

$$\mathrm{var}(V) = 2\int_0^a x^2 \frac{a-x}{a^2}\,dx = \frac{a^2}{6}$$

(iii)

$$E\{V^2/R\} = (E\{V\})^2 + \mathrm{var}(V) = V_0^2 + \frac{a^2}{6}$$

(iv)

$$\frac{E\{V^2\}}{V_0^2} = 1 + \frac{a^2}{6V_0^2} \qquad\qquad \frac{a^2}{6V_0^2} = 0.2^2/6 = 0.67\%$$

ENGR352 Problem Set 06 SOLUTION

1.

(i)

$$E(\tau) = \int_0^\infty \tau e^{-\tau}\, d\tau = 1$$

(ii)

$$p(\tau|\tau \geq \tau_0) = \frac{p(\tau)}{1 - F(\tau_0)} \qquad \tau \geq \tau_0$$

where the CDF of τ is

$$F(\tau_0) = \int_0^{\tau_0} e^{-\tau} d\tau = 1 - e^{-\tau_0}$$

Thus

$$P\{\tau \geq \tau_0\} = 1 - F(\tau_0) = e^{-\tau_0}$$

and

$$p(\tau|\tau \geq \tau_0) = \frac{p(\tau)}{e^{-\tau_0}} = e^{-(\tau-\tau_0)} \qquad \tau \geq \tau_0$$

Then use the transformation

$$\tau_1 = \tau - \tau_0$$

to get

$$p_{\tau_1}(\tau_1) = p_\tau(\tau_1 + \tau_0|\tau_1 + \tau_0 \geq \tau_0) = e^{-\tau_1} \qquad \tau_1 \geq 0$$

i.e., exactly the same as $p(\tau)$! This is because the exponential distribution is "memoryless".

(iii) In view of the last observation above, one has

$$E(\tau_1) = 1$$

(iv) The exponential distribution is a questionable model for this problem because of its memoryless property. Knowing the machine did not break down up to $\tau_0 = 1$ does not change (with this model) the distribution of the remaining life, which doesn't seem to be reasonable.

2.

(i) Using integration by parts

$$E(\tau) = \int_0^\infty \tau p(\tau)\, d\tau = \sqrt{\frac{\pi}{2}} a = 1$$

because $a = \sqrt{2/\pi}$.

(ii)

$$p(\tau|\tau \geq \tau_0) = \frac{p(\tau)}{1 - F(\tau_0)} \qquad \tau \geq \tau_0$$

Since

$$F(\tau_0) = \int_0^{\tau_0} \frac{\tau}{a^2} e^{-\frac{\tau^2}{2a^2}}\, d\tau = 1 - e^{-\frac{\tau_0^2}{2a^2}}$$

$$p(\tau|\tau \geq \tau_0) = \frac{p(\tau)}{e^{-\frac{\tau_0^2}{2a^2}}} = \frac{\tau}{a^2} e^{-\frac{\tau^2 - \tau_0^2}{2a^2}} \qquad \tau \geq \tau_0$$

Then use the transformation $\tau_1 = \tau - \tau_0$ to get $p_{\tau_1}(\tau_1)$

$$p_{\tau_1}(\tau_1) = p_\tau(\tau_1 + \tau_0|\tau_1 + \tau_0 \geq \tau_0) = \frac{\tau_1 + \tau_0}{a^2} e^{-\frac{(\tau_1+\tau_0)^2 - \tau_0^2}{2a^2}} \qquad \tau_1 \geq 0$$

(iii) We have (using integration by parts)

$$\begin{aligned}
E(\tau_1) &= \int_0^\infty \tau_1 p_{\tau_1}(\tau_1)\, d\tau_1 \\
&= \int_0^\infty \tau_1 \frac{\tau_1 + \tau_0}{a^2} e^{-\frac{(\tau_1+\tau_0)^2 - \tau_0^2}{2a^2}} d\tau_1 \\
&= \sqrt{2\pi} a e^{\frac{\tau_0^2}{2a^2}} [1 - G(\frac{\tau_0}{a})] \\
&= 2e^{\pi/4}[1 - G(\sqrt{\pi/2})] \\
&\approx 0.4608
\end{aligned}$$

where G is the cumulative standard Gaussian distribution.

(iv) The Rayleigh distribution is more reasonable than the exponential one for this problem. Knowing the machine did not break down up to $\tau_0 = 1$ should yield the remaining life shorter than at the initial time. This fits the real world experience better.

ENGR352 Problem Set 07 SOLUTION

1.

(i) From (1.4.14-18)

$$\mathrm{var}(x|y) = \sigma_x^2 - (\rho\,\sigma_x\sigma_y)^2(\sigma_y^2)^{-1} = (1-\rho^2)\sigma_x^2$$

Thus

$$\rho = \pm 1/\sqrt{2}$$

(ii) The smallest achievable $\mathrm{var}(x|y)$ is 0 and the values of ρ yield this are ± 1.

(iii) In this case x and y are linearly dependent, i.e.,

$$x = cy$$

with $c > 0$ if $\rho = 1$ and $c < 0$ if $\rho = -1$. Note that the actual value of c is irrelevant, only its sign matters.

2.

(i) From (1.4.19-19) one has

$$e^{-10a} = 0.1$$

which yields

$$a = -\frac{\ln 0.1}{10} = \frac{\ln 10}{10}\,\mathrm{s}^{-1}$$

(ii) From (14.19-21) one has

$$S_0 = 2aR_{xx}(0) = 2\,\frac{\ln 10}{10}\,5 = \ln 10\,\mathrm{V}^2/\mathrm{s}$$

(iii) The units of $n(t)$ are the same as those of $\dot{x}$, i.e., V/s. Thus the units of its power are $(\mathrm{V/s})^2$ (the convention in psd is that the square of the waveform is its power, i.e., as if it was a voltage across a unit resistance). The units of its psd are $(\mathrm{V/s})^2/\mathrm{Hz}=\mathrm{V}^2/\mathrm{s}$ since $\mathrm{Hz}=\mathrm{s}^{-1}$.

ENGR352 Problem Set 08 SOLUTION

1.

(i) The steady state probability vector will be the solution of the equation (written for convenience in transposed form)

$$\mu' = \mu' \Pi$$

i.e., the eigenvector corresponding to the unity eigenvalue of Π (ICBS that a probability transition matrix, whose rows sum up to unity, is guaranteed to have a unity eigenvalue).

This yields

$$\mu'(I - \Pi) = \mu' \left(I - \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \right) = \mu' \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix} = 0$$

which yields one equation for the elements μ_i, $i = 1, 2$ of μ. The other one is

$$\mu_1 + \mu_2 = 1$$

The solution is

$$\mu = [2/3,\ 1/3]'$$

(ii) The ratio of the time this system spends in state 1 vs. in state 2 is

$$\mu_1/\mu_2 = 2/1$$

2.

(i) The mean is, due to symmetry, x_0. The variance of x_i is

$$\sigma_i^2 = \int_{x_0-\epsilon}^{x_0+\epsilon} (x - x_0)^2 \, dx = \epsilon^2/3$$

(ii) What we need is

$$P\{L \in [999.905\text{m},\ 1000.095\text{m}]\}$$

Using the Central Limit Theorem,

$$L \sim \mathcal{N}(N x_0, \sigma_L^2)$$

where the s.d. of L is

$$\sigma_L = \sqrt{N\sigma_i^2} = \sqrt{100 \cdot (10^{-2})^2/3} = 0.058\text{m}$$

Thus, the length of the allowed interval for L is on each side of its mean, in units of its s.d. 0.095/0.058=1.64, and consequently, the above probability is 90% (the 5% upper tail and 5% lower tail are excluded — see Table 1.5.4-1, last row).

ENGR352 Problem Set 09 SOLUTION

1. Using Table 1.5.4-1:

(i) [0, 448]

(ii) [346, 457]

(iii) Dividing the above by $n/4 = 100$ yields the interval [3.46, 4.57]

Relative to $E\{y_{400}\} = 4$, the interval is [0.865, 1.1425].

(iv) [724, 880] divided by $n/4 = 200$ yields [3.62, 4.40].

Relative to $E\{y_{800}\} = 4$, the interval is [0.905, 1.10].

(v) $(1.1425 - 0.865)/(1.10 - 0.905) \approx 1.42 \approx \sqrt{2} = \sqrt{800/400}$

(vi) The two relative interval lengths are inversely proportional to the corresponding n (this is a consequence of the CLT, which will become clear later in Sec. 5.4).

ENGR352 Problem Set 10 SOLUTION

1.

(i)

$$\hat{x}^{\mathrm{MAP}} = \frac{100}{100+400}5 + \frac{400}{100+400}10 = 9$$

(ii)

$$\sigma_1^2 = \frac{100 \cdot 400}{100+400} = 80$$

(iii)

$$p(x|z) = \frac{1}{\sqrt{2\pi 80}} e^{-\frac{(x-9)^2}{2 \cdot 80}}$$

2.

(i) The likelihood function of a is

$$p(z|a) = \frac{z}{a^2}\, e^{-\dfrac{z^2}{2a^2}}$$

The likelihood equation for a is

$$\frac{d}{da}\left[\frac{z}{a^2}\, e^{-\dfrac{z^2}{2a^2}}\right] = 0$$

which yields

$$a = z/\sqrt{2}$$

(ii) The likelihood function of a is

$$p(z_1, \ldots, z_n|a) = \frac{\prod_{i=1}^n z_i}{a^2}\, e^{-\dfrac{\sum_{i=1}^n z_i^2}{2a^2}}$$

The likelihood equation for a is

$$\frac{d}{da}\left[\frac{\prod_{i=1}^n z_i}{a^2}\, e^{-\dfrac{\sum_{i=1}^n z_i^2}{2a^2}}\right] = 0$$

which yields

$$a = \sqrt{\frac{\sum_{i=1}^n z_i^2}{2}}$$

ENGR352 Problem Set 11 SOLUTION

Problem 2-4 from the text.

1. Let

$$\hat{\sigma}_k^2 \triangleq [\hat{\sigma}(k,n)]^2 \triangleq \frac{1}{n} f_1$$

Set

$$\frac{d}{dn} E[(\hat{\sigma}_k^2 - \sigma_0^2)^2] = 0$$

or

$$E[(\hat{\sigma}_k^2 - \sigma_0^2) f_1] = 0$$

or

$$E[\frac{1}{n} f_1^2] = E[\sigma_0^2 f_1] = \sigma_0^2 E[f_1]$$

Thus

$$n = \frac{E[f_1^2]}{E[f_1]\sigma_0^2}$$

Following some lengthy derivations and using a Gaussian assumption (for the calculation of the fourth moment) yields

$$n = \frac{\sigma_0^4(k^2-1)}{\sigma_0^4(k-1)} = k+1$$

2. Yes. Reason: the optimal estimator $[\hat{\sigma}(k)]^2$ in part (1) (with $n = k+1$) is biased:

$$E\{[\hat{\sigma}(k)]^2\} = \frac{k-1}{k+1} E\{\cdot\} = \frac{k-1}{k+1}\sigma^2$$

where $\{\cdot\}$ is the unbiased sample variance (2.5.3-8).

ENGR352 Problem Set 12 SOLUTION

Problem 2-12 from the text.

Assuming a $\frac{1}{100}\chi^2_{100}$ distribution for the ratio of the sample variance and the CRLB variance, its 95% probability region is $[0.742, 1.30]$, i.e., it is a perfectly reasonable outcome if this estimator is efficient.

Problem 2-14 from the text.

$$\sigma_{(\hat{\sigma}^{\mathrm{ML}})^2} = \sigma^2\sqrt{2/N}$$

The one-sided 95% probability region about the borderline value should start at 80:

$$100 - 1.64 \cdot 100\sqrt{2/N} = 80$$

which yields

$$N = \frac{2 \cdot 1.64^2}{0.2^2} \approx 135$$

ENGR352 Problem Set 13 SOLUTION

1.

(i)

$$E[z] = \bar{z} = E[Hx + w] = HE[x] + E[w] = H\bar{x} + 0 = H\bar{x}$$

(ii)

$$\begin{aligned}
P_{zz} &= E[(z-\bar{z})(z-\bar{z})'] \\
&= E[(Hx + w - H\bar{x})(Hx + w - H\bar{x})'] = E\{[H(x-\bar{x}) + w][H(x-\bar{x}) + w]'\} \\
&= E\{[H(x-\bar{x})][H(x-\bar{x})]' + ww'] = HE\{(x-\bar{x})(x-\bar{x})\}H' + E[w^2] \\
&= HP_{xx}H' + \sigma_w^2 = [1\ 1] \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 = 4
\end{aligned}$$

where the the crossterm in line 3 is not shown because it is zero.

Similarly,

$$P_{xz} = E[(x-\bar{x})(z-\bar{z})'] = P_{xx}H' = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$

These results are independent of the initial estimate of x.

(iii)

$$\begin{aligned}
P_{xx|z} &= P_{xx} - P_{xz}P_{zz}^{-1}P_{xz} \\
&= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} \frac{1}{4} [1.5\ \ 1.5] \\
&= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} - \begin{bmatrix} 0.5625 & 0.5625 \\ 0.5625 & 0.5625 \end{bmatrix} \\
&= \begin{bmatrix} 0.4375 & -0.0625 \\ -0.0625 & 0.4375 \end{bmatrix}
\end{aligned}$$

(iv)

$$\sqrt{\begin{vmatrix} 1 & 0.5 \\ 0.5 & 1 \end{vmatrix}} = 0.866$$

$$\sqrt{\begin{vmatrix} 0.4375 & -0.0625 \\ -0.0625 & 0.4375 \end{vmatrix}} = 0.442$$

i.e., a reduction of almost 50%.

ENGR352 Computer Project 14 SOLUTION

Localization based on range measurements

(i) To obtain the ILS estimator for this problem, linearize the measurement equation around an estimate $\hat{x}$

$$\begin{aligned}
d(x, x_{S_i}) &= \|x - x_{S_i}\| \\
&= \sqrt{(\xi - \xi_{S_i})^2 + (\eta - \eta_{S_i})^2 + (\zeta - \zeta_{S_i})^2} \\
&\approx d(\hat{x}, x_{S_i}) + (\nabla_x d(x, x_{S_i})|_{x=\hat{x}})'(x - \hat{x}) \\
&\approx \|\hat{x} - x_{S_i}\| + \frac{(\hat{x} - x_{S_i})'(x - \hat{x})}{\|\hat{x} - x_{S_i}\|} \\
&\triangleq \hat{z}_i(\hat{x}) + h(\hat{x}, x_{S_i})'(x - \hat{x}) \qquad i = 1, \ldots, 4
\end{aligned}$$

where

$$h(\hat{x}, x_{S_i})' \triangleq \frac{(\hat{x} - x_{S_i})'}{\|\hat{x} - x_{S_i}\|}$$

Following the j-th iteration of the ILS, the linearized system can be written as

$$\triangle z_i(\hat{x}_j) \triangleq z_i - \hat{z}_i(\hat{x}_j) = h(\hat{x}_j, x_{S_i})'(x - \hat{x}_j) + w_i \qquad i = 1, \ldots, 4$$

Using the 4 available measurements, one has equation:

$$\begin{aligned}
\triangle z(\hat{x}_j) &= [\triangle z_1(\hat{x}_j) \ldots \triangle z_4(\hat{x}_j)]' \\
&= \begin{bmatrix} h(\hat{x}_j, x_{S_1})' \\ \vdots \\ h(\hat{x}_j, x_{S_4})' \end{bmatrix} (x - \hat{x}_j) + w \\
&\triangleq H(\hat{x}_j)(x - \hat{x}_j) + w
\end{aligned}$$

where w is the four dimensional vector of the measurement noises.

Thus, using the ILS method, the next estimate $\hat{x}_{j+1}$ is

$$\hat{x}_{j+1} = \hat{x}_j + [H(\hat{x}_j)' R^{-1} H(\hat{x}_j)]^{-1} H(\hat{x}_j)' R^{-1} \triangle z(\hat{x}_j)$$

where

$$R = E[ww'] = \sigma^2 I$$

is the measurement noise covariance matrix. Consequently,

$$\hat{x}_{j+1} = \hat{x}_j + [H(\hat{x}_j)' H(\hat{x}_j)]^{-1} H(\hat{x}_j)' \triangle z(\hat{x}_j)$$

(ii) For this problem's geometry, the algorithm usually converges after the 3rd iteration. The MSE and RMSE values obltained from $N = 100$ runs are shown in the table below.

(iii) The covariance of the estimate is

$$P = [H(\hat{x}_n)' R^{-1} H(\hat{x}_n)]^{-1}$$

where n is the last iteration. The diagonal terms of this matrix are shown in the table below under TMSE (theoretical MSE).

	MSE (m^2)	RMSE (m)	TMSE (m^2)
x	286	17	307
y	287	17	310
z	429	21	400

From (2.6.3-6), the sample MSE from N runs should be (with 95% probability) within

$$2\sqrt{2P_{ii}^2/N} \approx 0.3P_{ii}$$

of the theoretical value P_{ii}, the diagonal terms of the calculated covariance of the estimate. The above sample MSEs satisfy this and thus they are statistically compatible.

ENGR352 Problem Set 15 SOLUTION

1.

$$
\begin{aligned}
P_{xx|z}^{-1} &= P(k+1)^{-1} = P(k)^{-1} + H'R(k)^{-1}H \\
&= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1^{-1} \; [1 \;\; 1] \\
&= \begin{bmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 7/3 & 1/3 \\ 1/3 & 7/3 \end{bmatrix}
\end{aligned}
$$

$$
P_{xx|z} = \begin{bmatrix} 7/3 & 1/3 \\ 1/3 & 7/3 \end{bmatrix}^{-1} = \begin{bmatrix} 7/16 & -1/16 \\ -1/16 & 7/16 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0.4375 & -0.0625 \\ -0.0625 & 0.4375 \end{bmatrix}
$$

ENGR352 Problem Set 16 SOLUTION

Problem 3-9 from the text.

$$\bar{y} = A\bar{x}_1 + B\bar{x}_2$$

$$P_{yy} = AP_{11}A' + AP_{12}B' + BP_{21}A' + BP_{22}B'$$

$$\bar{z} = C\bar{y} + D\bar{w}$$

$$P_{zz} = CP_{yy}C' + DP_wD'$$

$$P_{yz} = P_{yy}C'$$

The solution is then given by Eqs. (3.3.2-10) and (3.3.2-12) with the above terms.

Problem 3-10 from the text.

Assuming a constant velocity and using LS, Eq. (3.5.2-12) yields

$$P_{22} \approx \frac{64^2}{8^3}\frac{12}{\frac{1}{6^2}} = 8 \cdot 12 \cdot 36 \approx 3600 = (60\,\mathrm{ft/min})^2$$

ENGR352 Problem Set 17 SOLUTION

Problem 3-11 from the text.

1. Using (3.5.1-19) with $k+1=n$ yields for the position variance

$$\sigma^2 \frac{2[2(n-1)+3]}{(n-1)n} = 4\sigma^2 \frac{n+0.5}{n(n-1)}$$

2.

$$\frac{\sigma^2}{n}$$

Problem 3-12 from the text.

1.

$$\bar{y}_1 = \mu_2 + \mu_3 \qquad \bar{y}_2 = \mu_2 - \mu_3$$

$$\mathrm{var}(y_1) = \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{23}^2 \qquad \mathrm{var}(y_2) = \sigma_{22}^2 + \sigma_{33}^2 - 2\sigma_{23}^2$$

$$\mathrm{cov}(x_1, y_1) = \sigma_{12}^2 + \sigma_{13}^2 \qquad \mathrm{cov}(x_1, y_2) = \sigma_{12}^2 - \sigma_{13}^2$$

$$E[x_1|y_1] = \mu_1 + \frac{\sigma_{12}^2 + \sigma_{13}^2}{\sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{23}^2}(y_1 - \bar{y}_1)$$

$$\mathrm{var}(x_1|y_1) = \sigma_{11}^2 - \frac{(\sigma_{12}^2 + \sigma_{13}^2)^2}{\sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{23}^2}$$

$$E[x_1|y_2] = \mu_1 + \frac{\sigma_{12}^2 - \sigma_{13}^2}{\sigma_{22}^2 + \sigma_{33}^2 - \sigma_{13}^2}(y_2 - \bar{y}_2)$$

$$\mathrm{var}(x_1|y_2) = \sigma_{11}^2 - \frac{(\sigma_{12}^2 - \sigma_{13}^2)^2}{\sigma_{22}^2 + \sigma_{33}^2 - \sigma_{13}^2}$$

The required pdfs are Gaussian with the above moments.

ENGR352 Problem Set 18 SOLUTION

Problem 4-3 from the text.

1. With $k > j$, (4.3.3-1) yields

$$x(k) - \bar{x}(k) = \left[\prod_{i=0}^{k-1-j} F(k-1-i) \right] [x(j) - \bar{x}(j)] + \sum_{i=j}^{k-1} (\cdot) v(i)$$

Since $v(i)$ is zero-mean and white,

$$v(i) \perp x(j) \qquad \text{or} \qquad v(i) \perp [x(j) - \bar{x}(j)] \qquad \forall i \geq j$$

it follows that

$$V_{xx}(k,j) = \left[\prod_{i=0}^{k-1-j} F(k-1-i) \right] P_{xx}(j) \qquad\qquad \forall k \geq j$$

2. Since, for a stable LTI system, all the eigenvalues of F have magnitude less than 1,

$$V_{xx}(k,j) \to 0 \qquad \text{for} \qquad k \gg j$$

This means that $x(k)$ and $x(j)$ tend to be uncorrelated, that is, the effect of $x(j)$ (i.e., transient) dies out in a stable system.

Problem 4-4 from the text.

Equations (4.3.4-10), (4.3.4-13) and the whiteness of v yield

$$\begin{aligned}
\text{var}[x(k)] &= \sum_i \sum_j \alpha^{k-i} \alpha^{k-j} E[(v(i) - \bar{v}(i))(v(j) - \bar{v}(j))'] \\
&= \sum_i (\alpha^2)^{k-i} E[(v(i) - \bar{v}(i))(v(i) - \bar{v}(i))'] \\
&= \frac{1-\alpha^{2k}}{1-\alpha^2} \sigma^2
\end{aligned}$$

1.

$$y(k) = (1-\alpha) \sum_{i=1}^{k} \alpha^{k-i} v(i) = (1-\alpha) x(k)$$

2.

$$\bar{y}(k) = (1-\alpha^k)\bar{v} \quad \stackrel{k\to\infty}{\longrightarrow} \quad \bar{v}$$
$$\text{var}[y(k)] = \frac{1-\alpha}{1+\alpha}(1-\alpha^{2k})\sigma^2 \quad \stackrel{k\to\infty}{\longrightarrow} \quad \frac{1-\alpha}{1+\alpha}\sigma^2$$

3.

$$z(k) = \frac{1-\alpha}{1-\alpha^k} x(k) \qquad \Longrightarrow \qquad \bar{z}(k) = \bar{v}$$
$$\text{var}[z(k)] = \frac{(1-\alpha)(1-\alpha^{2k})}{(1+\alpha)(1-\alpha^k)^2}\sigma^2 \quad \stackrel{k\to\infty}{\longrightarrow} \quad \frac{1-\alpha}{1+\alpha}\sigma^2$$

Thus $y(k)$ is only an asymptotically unbiased estimator of $\bar{v}$ while $z(k)$ is an unbiased one. For large k they have the same variance, i.e., they are practically the same.

4.

$$N_e = \frac{1}{1-\alpha} = 10 \qquad \Longrightarrow \qquad \alpha = 0.9$$

ENGR352 Problem Set 19 SOLUTION

Problem 4-8 from the text.

Since the eigenvalues of this matrix are 0, 0, $\pm j\Omega$, use of the interpolating polynomial method is probably the simplest.

The series expansion method is also simple by noticing that the terms in each element are simple modification of the expansion of a $\sin\Omega t$ or $\cos\Omega t$.

ENGR352 Problem Set 21 SOLUTION

1.

(i) The state estimation error $\tilde{x}(k+1|k+1)$ can be written in terms of the previous prediction error $\tilde{x}(k+1|k)$ (for simplicity, for time invariant F, H, and W) as

$$\begin{aligned}
\tilde{x}(k+1|k+1) &\stackrel{\Delta}{=} x(k+1) - \hat{x}(k+1|k+1) \\
&= x(k+1) - \hat{x}(k+1|k) - W[z(k+1) - H\hat{x}(k+1|k)] \\
&= \tilde{x}(k+1|k) - W[Hx(k+1) + w(k+1) - H\hat{x}(k+1|k)] \\
&= \tilde{x}(k+1|k) - WH\tilde{x}(k+1|k) - Ww(k+1) \\
&= [I - WH]\tilde{x}(k+1|k) - Ww(k+1)
\end{aligned}$$

Multiplying the above with its transpose and taking its expectation yields (5.2.3-18). The cross term vanishes due to the whiteness of the (zero-mean) measurement noise. Note that, in the above, the gain W is arbitrary.

(ii) Using in the above

$$\tilde{x}(k+1|k) \stackrel{\Delta}{=} x(k+1) - \hat{x}(k+1|k) = F\tilde{x}(k|k) + v(k)$$

yields

$$\begin{aligned}
\tilde{x}(k+1|k+1) &= [I - WH][F\tilde{x}(k|k) + v(k)] - Ww(k+1) \\
&= [I - WH]F\tilde{x}(k|k) + [I - WH]v(k) - Ww(k+1)
\end{aligned}$$

(iii)

$$P(k+1|k+1) = [I - WH]FP(k|k)F'[I - WH]' + [I - WH]FQ(k)F'[I - WH]' + WR(k+1)W'$$

(iv) The matrix $[I - WH]F$ should have all its eigenvalues inside the unit circle.

ENGR352 Problem Set 22 SOLUTION

1. *Problem 5-1 from the text.*

1. Since the system is completely controllable and observable, there exists a unique $P_\infty > 0$ such that

$$P_\infty = \lim_{k\to\infty} P(k|k) = \lim_{k\to\infty} P(k+1|k+1) = \frac{rf^2 P_\infty + rq}{h^2 f^2 P_\infty + h^2 q + r}$$

$$P_\infty = \frac{-(r - rf^2 + qh^2) + \sqrt{(r - rf^2 + qh^2)^2 + 4rqh^2 f^2}}{2h^2 f^2} > 0$$

2. Note that the noise standard deviations are $\sigma_v = \sqrt{q}$, $\sigma_w = \sqrt{r}$.

3. Since

$$x(0) = \frac{1}{h} z(0) - \frac{1}{h} w(0)$$

we have

$$P(0|0) = E[(x(0) - z(0)/h)^2] = \frac{r}{h^2}$$

Suitable care must be exercised when the first sampling time is zero.

4. For a sample output of one run see Table 1. Note the convergence of the variance. The normalized state errors and the normalized innovations are, in magnitude, less than 2, most of the time; however, while the latter are white, the former appear (and are) correlated.

5. $P_\infty = 0.0951$

6. See Table 2 for $P(0|0) = 0$, and 10. Note that the steady-state value is independent of the initial variance.

7. Change the initial variance given to the filter *and* generate new initial estimate with the new variance.

2. *Computer Application 5-2 from the text.*

1. For $\rho = -0.5$

$$\begin{bmatrix} 3.6843 & 0.8238 \\ 0.8238 & 1.2866 \end{bmatrix}$$

2. For $\rho = 0$

$$\begin{bmatrix} 6.0218 & 1.6557 \\ 1.6557 & 1.7032 \end{bmatrix}$$

3. For $\rho = 0.5$

$$\begin{bmatrix} 6.4886 & 2.2403 \\ 2.2403 & 1.9260 \end{bmatrix}$$

The reason the negative correlation between the measurement errors in position and velocity is the best is that this counters the positive correlation between the estimation errors in position and velocity due to the dynamic equation.

k	$P(k\|k)$ for $P(0\|0) = 0$	$P(k\|k)$ for $P(0\|0) = 1$	$P(k\|k)$ for $P(0\|0) = 10$
1	0.0099	0.5025	0.9092
2	0.0195	0.3388	0.4789
3	0.0287	0.2586	0.3284
4	0.0372	0.2117	0.2528
5	0.0451	0.1815	0.2081
6	0.0522	0.1607	0.1791
7	0.0586	0.1458	0.1590
8	0.0642	0.1348	0.1446
9	0.0691	0.1265	0.1339
10	0.0733	0.1201	0.1258
11	0.0769	0.1151	0.1195
12	0.0799	0.1112	0.1147
13	0.0825	0.1081	0.1109
14	0.0847	0.1056	0.1078
15	0.0865	0.1036	0.1054
16	0.0880	0.1020	0.1035
17	0.0892	0.1008	0.1019
18	0.0903	0.0997	0.1006
19	0.0911	0.0989	0.0996
20	0.0919	0.0982	0.0988
21	0.0924	0.0976	0.0981
22	0.0929	0.0972	0.0976
23	0.0933	0.0968	0.0971
24	0.0936	0.0965	0.0968
25	0.0939	0.0962	0.0965
26	0.0941	0.0960	0.0962
27	0.0943	0.0959	0.0960
28	0.0945	0.0957	0.0959
29	0.0946	0.0956	0.0957
30	0.0947	0.0955	0.0956
31	0.0948	0.0955	0.0955
32	0.0948	0.0954	0.0955
33	0.0949	0.0953	0.0954
34	0.0949	0.0953	0.0953
35	0.0950	0.0953	0.0953
36	0.0950	0.0952	0.0953
37	0.0950	0.0952	0.0952
38	0.0950	0.0952	0.0952
39	0.0951	0.0952	0.0952
40	0.0951	0.0952	0.0952
41	0.0951	0.0952	0.0952
42	0.0951	0.0952	0.0952
43	0.0951	0.0952	0.0952
44	0.0951	0.0951	0.0952
45	0.0951	0.0951	0.0951
46	0.0951	0.0951	0.0951
47	0.0951	0.0951	0.0951
48	0.0951	0.0951	0.0951
49	0.0951	0.0951	0.0951
50	0.0951	0.0951	0.0951

Table 2. Convergence of the Riccati equation from different initial conditions.

ENGR352 Problem Set 23 SOLUTION

Problem 5-14 from the text.

1.

$$S = \text{diag}(5,\ 16) + \text{diag}(4,\ 9) = \text{diag}(9,\ 25)$$

$$\text{NIS} = \nu' S^{-1} \nu \leq \chi_2^2(95\%) = 6$$

2.

(a) NIS=2<6, accepted

(b) NIS=8>6, rejected

(c) NIS=13>6, rejected

3. $\chi_2^2(99\%) = 9.2$

(a) NIS=2<9.2, accepted

(b) NIS=8<9.2, accepted

(c) NIS=13>9.2, rejected

Problem 5-15 from the text.

The LS estimate and covariance of the parameter vector for a constant acceleration model are given in (3.5.2-15) and (3.5.2-14), respectively, where k is the number of measurements (3 in the current problem). However, they are for the center point, i.e., $i = 2$. Using the transformation equations (3.5.3-1)–(3.5.3-3) with $t = T$, yields

$$P(t = T|k = 3) = \begin{bmatrix} 1 & \frac{3}{2T} & \frac{1}{T^2} \\ \frac{3}{2T} & \frac{13}{6T^2} & \frac{6}{T^3} \\ \frac{1}{T^2} & \frac{6}{T^3} & \frac{6}{T^4} \end{bmatrix} \sigma^2$$

$$\hat{x}(t = T|k = 3) = \begin{bmatrix} z(3) \\ \frac{z(1)-4z(2)+3z(3)}{2T} \\ \frac{z(1)-2z(2)+z(3)}{T^2} \end{bmatrix}$$

ENGR352 Problem Set 24 SOLUTION

Problem 5-16 from the text.

Let

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Hx + w$$

where, with I denoting the identity matrix of the dimension of x,

$$H = \begin{bmatrix} I \\ I \end{bmatrix} \qquad w = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

and

$$Eww' = P$$

Then the likelihood function of x is

$$p(z|x) = \mathcal{N}(z; Hx, P)$$

The MLE of x is

$$\hat{x} = (H'P^{-1}H)^{-1}H'P^{-1}z = (P_{22} - P_{21})T^{-1}\hat{x}_1 + (P_{11} - P_{12})T^{-1}\hat{x}_2$$

where the inversion of a partitioned matrix has been used.

$$T \triangleq P_{11} - P_{12} + P_{22} - P_{21}$$

and

$$\text{cov}(\hat{x}) = (H'P^{-1}H)^{-1} = P_{22} - [P_{22} - P_{21}][P_{11} - P_{12} + P_{22} - P_{21}]^{-1}[P_{22} - P_{12}]$$

Problem 5-17 from the text.

1. The filter updated state variance is in steady state (from problem 5-1)

$$P_\infty = \frac{-q + \sqrt{q^2 + 4rq}}{2}$$

and the gain is

$$W_\infty = P_\infty / r = \frac{-q + \sqrt{q^2 + 4rq}}{2r}$$

2. Let

$$\mu = q/r$$

Then

$$W_\infty = \frac{-\mu + \sqrt{\mu^2 + 4\mu}}{2}$$

3. The filter gain is

$$W_{\infty_f} = \frac{-q_f + \sqrt{q_f^2 + 4rq_f}}{2r}$$

and the equation for the s.s. estimation MSE is

$$P_\infty = (1 - W_{\infty_f})^2 (P_\infty + q) + W_{\infty_f}^2 r$$

which yields

$$P_\infty = \frac{(1 - W_{\infty_f})^2 q + W_{\infty_f}^2 r}{1 - (1 - W_{\infty_f})^2}$$

4.

$$|1 - W_{\infty_f}| < 1$$

ENGR352 Problem Set 26 SOLUTION

Problem 6-1 from the text.

1. If the acceleration $\ddot{\xi}(k) = v(k)$ is a zero-mean white Gaussian noise, the velocity $\dot{\xi}(k)$ is then its integral, i.e., a (discrete-time) Wiener process

$$\dot{\xi}(k) = \dot{\xi}(0) + T \sum_{i=0}^{k-1} v(i)$$

and, therefore,

$$\dot{\xi}(k) \sim \mathcal{N}[\dot{\xi}(0), kT^2\sigma_v^2]$$

2. The range of the velocity after k sampling periods is

$$P\{\dot{\xi}(k) \in [\dot{\xi}(0) - 1.96\sqrt{k}T\sigma_v, \dot{\xi}(0) + 1.96\sqrt{k}T\sigma_v]\} = 95\%$$

3. The lower bound of the above confidence region is $10 - 2 \times 5 = 0$. Therefore, this is an acceptable change and it is not a sign of bias in the random numbers. Thus, while the average change in the velocity is zero, such a process noise can significantly change it in a given sample trajectory (run).

4. In the third order model the velocity is the integral of a Wiener process and even larger changes can occur.

ENGR352 Problem Set 27 SOLUTION

Problem 6-8 from the text.

1. In this problem $\lambda = 2$. As in the previous problem, one has

$$\alpha = \frac{-\lambda^2 + \sqrt{\lambda^4 + 16\lambda^2}}{8}$$

which yields $\alpha = 0.62$.

2. They are α and $\sqrt{\alpha}$, respectively.

3. The Joseph form equation for the estimate's variance P is

$$P = (1 - W)^2(P + 1) + W^2$$

which yields $P = 4.3 > r$, i.e., no reduction but magnification!

4. $|1 - W| < 1$, i.e., $0 < W < 2$.

5. P diverges.

Problem 6-9 from the text.

1.

$$\hat{x}(k+1) = (1-\alpha)\hat{x}(k) + \alpha z(k+1)$$

$$\hat{x}(k+1) - x(k+1) = (1-\alpha)\hat{x}(k) - x(k) + \alpha[x(k) + w(k+1)]$$

$$-\tilde{x}(k+1) = -(1-\alpha)\tilde{x}(k) + \alpha w(k+1)$$

The s.s. estimation variance is then

$$P = (1-\alpha)^2 P + \alpha^2 r$$

$$P = \frac{\alpha}{2-\alpha} r$$

2. Denoting by x_v the state driven by the constant v, one has

$$x_v(k+1) = x_v(k) + v$$

The estimate of x_v in the absence of measurement noise is then

$$\hat{x}_v(k+1) = (1-\alpha)\hat{x}_v(k) + \alpha x_v(k+1)$$

with error

$$\hat{x}_v(k+1) - x_v(k+1) = (1-\alpha)\hat{x}_v(k) - x_v(k) - v + \alpha[x_v(k) + v]$$

$$-\tilde{x}_v(k+1) = -(1-\alpha)\tilde{x}_v(k) - (1-\alpha)v$$

$$b = (1-\alpha)b + (1-\alpha)v$$

$$b = \frac{1-\alpha}{\alpha} v$$

3. By superposition, the MSE is

$$M = P + b^2$$

ENGR352 Problem Set 28 SOLUTION

Problem 10-1 from the text.

The estimate of the location of the maximum is

$$\hat{x}_m = f(x_1, x_2, x_3, z_1, z_2, z_3)$$

with true value

$$x_m = f(x_1, x_2, x_3, y_1, y_2, y_3)$$

Using a first order expansion one has

$$\hat{x}_m = x_m + \sum_{i=1}^{3} \frac{\partial f}{\partial z_i}(z_i - y_i) = x_m + (\nabla_z f)' w$$

The error in the location of the maximum is in terms of the noises

$$\tilde{x}_m = (\nabla_z f)' w$$

1. To find the expression of $\hat{x}_m$ consider the parabolic approximation

$$z(x) = ax^2 + bx + c$$

which leads to

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Setting the differential to zero yields

$$\hat{x}_m = -\frac{b}{2a} \qquad \Longrightarrow \qquad f = -\frac{b}{2a}$$

2. The error is, in terms of the noise vector w,

$$\tilde{x}_m = (\nabla_z f)' w = \frac{b(\nabla_z a)' - a(\nabla_z b)'}{2a^2} w \triangleq u' w$$

and, thus, the MS error (variance) is

$$E[\tilde{x}_m^2] = u' P u$$

3. In this case it is convenient to write the parabolic approximation

$$y = a(x-n)^2 + b(x-n) + c$$

in the neighborhood of $x = n$. These coefficients, which are different from those in part (1), are obtained from the equations (obtained by substituting x_i into the above equation)

$$a(-1)^2 + b(-1) + c = z_1$$

$$c = z_2$$

$$a(1)^2 + b(1) + c = z_3$$

as

$$a = (z_1 + z_3 - 2z_2)/2 \qquad b = (z_3 - z_1)/2$$

The maximum of the parabola is at

$$\hat{x} = n - \frac{b}{2a} = n + \frac{z_3 - z_1}{2(2z_2 - z_1 - z_3)} \triangleq n + \frac{\alpha_1}{\alpha_2}$$

The location of the maximum based on the exact values of the function is

$$x = n + \frac{y_3 - y_1}{2(2y_2 - y_1 - y_3)}$$

and, using a first order series expansion,

$$\hat{x} = x + \sum_{i=1}^{3} \frac{\partial}{\partial z_i}\left(\frac{\alpha_1}{\alpha_2}\right)(z_i - y_i) \triangleq x + \sum_{i=1}^{3} u_i w_i$$

where

$$u_1 = -\frac{1}{\alpha_2} + \frac{2\alpha_1}{\alpha_2^2} \qquad u_2 = -\frac{4\alpha_1}{\alpha_2^2} \qquad u_3 = \frac{1}{\alpha_2} + \frac{2\alpha_1}{\alpha_2^2}$$

are the (first order) sensitivity functions of the location of the maximum w.r.t. the observation errors w_i.

For the given numbers one has

$$\alpha_1 = 0 \qquad a_2 = 0.8 \qquad \hat{x} = n$$

$$u_1 = -1.25 \qquad u_2 = 0 \qquad u_3 = 1.25 \qquad \sigma^2 = 0.03125$$

Problem 10-2 from the text.

$$\sqrt{1-\alpha} \approx 1 - \frac{\alpha}{2} \qquad \Longrightarrow \qquad (100\alpha/2)\%$$

ENGR352 Problem Set 29 SOLUTION

Problem 10-7 from the text.

1.

$$\hat{v} = \sqrt{\hat{x}_3^2 + \hat{x}_4^2}$$

2. Using first order series expansion

$$\tilde{v} = \frac{\hat{x}_3}{\hat{v}}\tilde{x}_3 + \frac{\hat{x}_4}{\hat{v}}\tilde{x}_4 \triangleq \tilde{x}_3 \cos\hat{\phi} + \tilde{x}_4 \sin\hat{\phi}$$

$$E\tilde{v}^2 = P_{33}(\cos\hat{\phi})^2 + P_{44}(\sin\hat{\phi})^2 + P_{34}\sin 2\hat{\phi}$$

3.

$$\theta = 90 - \phi = 90 - \tan^{-1}\frac{x_4}{x_3}$$

$$\hat{\theta} = 90 - \tan^{-1}\frac{\hat{x}_4}{\hat{x}_3}$$

4. Using first order series expansion

$$\tilde{\theta} = -\left[\frac{\partial}{\partial x_4}\tan^{-1}\frac{\hat{x}_4}{\hat{x}_3}\right]_{x=\hat{x}}\tilde{x}_4 - \left[\frac{\partial}{\partial x_3}\tan^{-1}\frac{\hat{x}_4}{\hat{x}_3}\right]_{x=\hat{x}}\tilde{x}_3 = \frac{-\hat{x}_3\tilde{x}_4 + \hat{x}_4\tilde{x}_3}{\hat{x}_3^2 + \hat{x}_4^2}$$

$$E[\tilde{\theta}^2] = \frac{\hat{x}_3^2 P_{44} - 2\hat{x}_3\hat{x}_4 P_{34} + \hat{x}_4^2 P_{33}}{[\hat{x}_3^2 + \hat{x}_4^2]^2}$$

5.

$$\sigma_{\tilde{\theta}} = \sqrt{E[\tilde{\theta}^2]}\frac{180}{\pi}$$

6.

$$\hat{v} = 14$$

$$E\tilde{v}^2 = 1 \cdot 0.5 + 1 \cdot 0.5 + 0.5 \cdot 1 = 2.5 = (1.58)^2$$

$$\hat{\theta} = 45^\circ$$

$$E[\tilde{\theta}^2] = \frac{100(1 - 1 + 1)}{200^2} = 0.0025\text{rad}^2 = (0.05\text{rad})^2 \approx (3^\circ)^2$$

Problem 10-8 from the text.

1. The true range is

$$r = \sqrt{R^2 - h^2}$$

Using a first order series expansion, the error in r is (approximately) zero mean and with variance

$$\sigma_r^2 \approx \left[\frac{\partial r}{\partial R}\right]^2 \sigma_R^2 + \left[\frac{\partial r}{\partial h}\right]^2 \sigma_h^2$$

The partials, to be evaluated at the measured/assumed values, are

$$\frac{\partial r}{\partial R} = \frac{R}{r}$$

$$\frac{\partial r}{\partial h} = -\frac{h}{r}$$

yield

$$\sigma_r^2 \approx \left[\frac{R}{r}\right]^2 \sigma_R^2 + \left[\frac{h}{r}\right]^2 \sigma_h^2$$

2.

$$r \approx R = 10^5 \qquad h/r = 1/50$$

$$\sigma_r^2 \approx 10^4 + \left[\frac{1}{50}\right]^2 10^6 \approx 10^4$$

3. The altitude error at this range has negligible effect. For $R = 10^4$ one has $h/r = 1/5$ and

$$\sigma_r^2 \approx 10^4 + \left[\frac{1}{5}\right]^2 10^6 \approx 5 \cdot 10^4$$

ENGR352 Problem Set 30 SOLUTION

Problem 10-10 from the text.

1.

$$x - x_1 = \sqrt{r^2 - y_1^2}$$

$$\frac{dx}{dr} = \frac{r}{\sqrt{r^2 - y_1^2}} = \frac{\sqrt{y_1^2 + (x_0 - x_1)^2}}{|x_0 - x_1|}$$

2. GDOP $\approx 10^5/10^4 = 10$

3. GDOP $\approx 10^6/10^6 = 1$

4. Let $\sigma_r = 1$. Then

$$p(\mathbf{r}|\mathbf{p}) = \mathcal{N}(\mathbf{r}; \psi(\mathbf{p}), I)$$

$$\nabla_{\mathbf{p}} \ln p(\mathbf{r}|\mathbf{p}) = -\nabla_{\mathbf{p}}\psi(\mathbf{p}, \mathbf{X})'[\mathbf{r} - \psi(\mathbf{p}, \mathbf{X})]$$

The FIM is then

$$J = E[\nabla_{\mathbf{p}} \ln p(\mathbf{r}|\mathbf{p})][\nabla_{\mathbf{p}} \ln p(\mathbf{r}|\mathbf{p})]' = \nabla_{\mathbf{p}}\psi(\mathbf{p}, \mathbf{X})'[\nabla_{\mathbf{p}}\psi(\mathbf{p}, \mathbf{X})']' = \left(\frac{\partial\psi}{\partial\mathbf{p}}\right)' \frac{\partial\psi}{\partial\mathbf{p}}$$

where $\frac{\partial\psi}{\partial\mathbf{p}}$ is the Jacobian of the measurement function.

Thus the covariance of the estimate is (asuming efficiency)

$$P_{\mathbf{p}} = J^{-1}$$

and the RMS position error is

$$\text{RMS}(\tilde{\mathbf{p}}) = \sqrt{\text{tr}(P_{\mathbf{p}})} = \sqrt{\text{tr}\left\{\left[\left(\frac{\partial\psi}{\partial\mathbf{p}}\right)' \frac{\partial\psi}{\partial\mathbf{p}}\right]^{-1}\right\}}$$

The components of ψ are

$$\psi_i = \sqrt{(\mathbf{p} - \mathbf{x_i})'(\mathbf{p} - \mathbf{x_i})}$$

$$\nabla_{\mathbf{p}}\psi_i = (\mathbf{p} - \mathbf{x_i})/\psi_i$$

$$\nabla_{\mathbf{p}}\psi' = [\nabla_{\mathbf{p}}\psi_1, \ldots, \nabla_{\mathbf{p}}\psi_n]$$

The FIM is

$$J = \sum_{i=1}^{n} (\nabla_{\mathbf{p}}\psi_i)' (\nabla_{\mathbf{p}}\psi_i) = \sum_{i=1}^{n} (\mathbf{p} - \mathbf{x_i})(\mathbf{p} - \mathbf{x_i})'/\psi_i^2$$

5.

$$J = \left\{ \begin{bmatrix} 10^{10} & -10^9 \\ -10^9 & 10^{10} \end{bmatrix} + \begin{bmatrix} 10^{10} & 10^9 \\ 10^9 & 10^{10} \end{bmatrix} \right\} (10^{10} + 10^8)^{-1}$$

$$J^{-1} \approx \begin{bmatrix} 50 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\sqrt{\operatorname{tr}(J^{-1})} \approx \sqrt{50} \approx 7$$

Problem 10-11 from the text.

1.

$$a_{\text{long}} = \frac{a'v}{\|v\|^2}\, v = \frac{\ddot{x}\dot{x} + \ddot{y}\dot{y}}{\dot{x}^2 + \dot{y}^2}\, [\dot{x}\ \dot{y}]'$$

$$\|a_{\text{long}}\| = \frac{\ddot{x}\dot{x} + \ddot{y}\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

2.

$$a_{\text{lat}} = a - a_{\text{long}}$$

$$\|a_{\text{lat}}\| = \sqrt{\|a\|^2 - \|a_{\text{long}}\|^2} = \frac{|\ddot{x}\dot{x} - \ddot{y}\dot{y}|}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

3.

$$\omega = \frac{\|a_{\text{lat}}\|}{\|v\|} = \frac{|\ddot{x}\dot{x} - \ddot{y}\dot{y}|}{\dot{x}^2 + \dot{y}^2}$$

4.

$$\sigma_\omega^2 = (\partial\omega/\partial\mathbf{x})\, P\, (\partial\omega/\partial\mathbf{x})'$$

ENGR352 Problem Set 32 SOLUTION

Problem 11-2 from the text.

This problem calls for the use of the mixture equations

1.

$$\hat{x} = E[x|z] = E[x|z, \alpha = 1]p_1 + E[x|z, \alpha = 0]p_2$$

where, from Subsections 2.3.2 and 2.4.3,

$$E[x|z, \alpha = 1] = \frac{x_0\sigma_0^{-2} + z\sigma_1^{-2}}{\sigma_0^{-2} + \sigma_1^{-2}} \triangleq \hat{x}_1$$

$$E[x|z, \alpha = 0] = \frac{x_0\sigma_0^{-2} + z\sigma_2^{-2}}{\sigma_0^{-2} + \sigma_2^{-2}} \triangleq \hat{x}_2$$

2. From Subsection 2.4.3

$$\begin{aligned} \mathrm{var}[x|z] &= \mathrm{var}[x|z, \alpha = 1]p_1 + \mathrm{var}[x|z, \alpha = 0]p_2 + p_1(\hat{x}_1 - \hat{x})^2 + p_2(\hat{x}_2 - \hat{x})^2 \\ &= \sigma_1^2 p_1 + \sigma_2^2 p_2 + p_1(\hat{x}_1 - \hat{x})^2 + p_2(\hat{x}_2 - \hat{x})^2 \end{aligned}$$

ENGR352 Problem Set 33 SOLUTION

Problem 11-6 from the text.

The pmf of the sojourn time in state i is

$$P\{\tau_i = k\} = (p_{ii})^{k-1}(1-p_{ii}) \qquad k = 1, 2, \ldots$$

$$\begin{aligned} E[\tau_i] &= \sum_{k=1}^{\infty} k(p_{ii})^{k-1}(1-p_{ii}) \\ &= (1-p_{ii})\sum_{k=0}^{\infty} k(p_{ii})^{k-1} \\ &= (1-p_{ii})\frac{d}{dp_{ii}}\left[\sum_{k=0}^{\infty}(p_{ii})^k\right] \\ &= (1-p_{ii})\frac{d}{dp_{ii}}\left[\frac{1}{1-p_{ii}}\right] \end{aligned}$$

Problem 11-7 from the text.

1.

$$\Pi = \begin{bmatrix} p_1 & 1-p_1 \\ p_1 & 1-p_1 \end{bmatrix}$$

2.

$$\tau_1 = \frac{1}{1-p_1}$$

$$p_1 = \frac{\tau_1 - 1}{\tau_1}$$

$$\tau_2 = \frac{1}{1-p_2} = \frac{1}{p_1} = \frac{\tau_1}{\tau_1 - 1}$$

ENGR352 Problem Set 35 SOLUTION

Problem 11-9 from the text.

(i) To obtain the IMM estimator, define the following four possible models based on the same WNA state model, but with different measurement equations, corresponding to the normal mode (designated as mode 1) and the failure modes (designated as modes 2,3,4):

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The Markov chain transition matrix entries are based on the duration of the cycles in the failure sequence. Using the above subscripts for the modes, the probability p_{11} is taken (this is subjective, other similar values can be also used) as 0.79 (approximation of 10/13); the jump probability to each failure mode (assuming them equal) in then 1/13≈0.07. The probabilities of staying in a failure mode, p_{ii}, $i = 2, 3, 4$, are taken as 0.9 (approximation of 10/11); the probabilities of jumping from each failure mode to the normal mode are taken as 2/3 of the remaining probability mass (0.06) and the probabilities of going into another failure mode are equal (0.02). Then

$$P = \begin{bmatrix} .79 & .07 & .07 & .07 \\ .06 & .9 & .02 & .02 \\ .06 & .02 & .9 & .02 \\ .06 & .02 & .02 & .9 \end{bmatrix}$$

Note: the use of Dynaest for this problem is limited since it allows only up to 5 “legs”, rather that 7 as are required in this problem. Hence the data for the problem was externally generated, and then filtered using Dynaest.

(ii) Based on 100 runs using the same failure sequence and independent noises and initial estimates for each run, the average number of samples needed to detect a mode change is 1, for both scenarios. The criterion to define a change of mode is that the mode probability exceeds the value 0.5. If instead of 0.5, we change this threshold to 0.8, then the number of samples needed to detect a mode change varies in scenario

A to about 2 samples (and as an extra drawback, the normal mode does not exceed this higher threshold, so it is not detected), while for scenario B it is still 1 sample. The cause of this difference is based on the fact that scenario A is just a random walk, so the position and velocity wander around zero, making it more difficult to detect the lack of some of these measuremente in a failure mode, but as the position variance grows in time, the later failures will be easier to detect than the earlier ones. On the other hand, scenario B represents a constant velocity model, for which the position grows linearly (plus noise) and the velocity is around 10, so in case of failure, the absence of any measurement is easily detected.

Figure 1: Mode probabilities for Scenario A.

Figure 2: Mode probabilities for Scenario B.

ENGR352 Computer Project 38 SOLUTION

Localization based on range measurements

(i) To obtain the ILS estimator, linearize the nonlinear part of the measurement equation around an estimate $\hat{x}$

$$\begin{aligned} h(x, b, x_{S_i}) &= d(x, x_{S_i}) + b \\ &= \|x - x_{S_i}\| + b \\ &= \sqrt{(\xi - \xi_{S_i})^2 + (\eta - \eta_{S_i})^2 + (\zeta - \zeta_{S_i})^2} + b \\ &\approx d(\hat{x}, x_{S_i}) + (\nabla_x d(x, x_{S_i})|_{x=\hat{x}})'(x - \hat{x}) + b \\ &= d(\hat{x}, x_{S_i}) + \hat{b} + (\nabla_x d(x, x_{S_i})|_{x=\hat{x}})'(x - \hat{x}) + b - \hat{b} \\ &= h(\hat{x}, \hat{b}, x_{S_i}) + d_x(\hat{x}, x_{S_i})'(x - \hat{x}) + b - \hat{b} \qquad i = 1, \ldots, 6 \end{aligned}$$

where

$$d_x(\hat{x}, x_{S_i})' \triangleq (\nabla_x d(x, x_{S_i})|_{x=\hat{x}})' = \frac{(\hat{x} - x_{S_i})'}{\|\hat{x} - x_{S_i}\|}$$

Let

$$\begin{aligned} \hat{z}_i(\hat{x}, \hat{b}) &\triangleq h(\hat{x}, \hat{b}, x_{S_i}) \\ &= \|\hat{x} - x_{S_i}\| + \hat{b} \end{aligned}$$

Then

$$\begin{aligned} z_i &= h(x, b, x_{S_i}) + w_i \\ &= h(\hat{x}, \hat{b}, x_{S_i}) + d_x(\hat{x}, x_{S_i})'(x - \hat{x}) + b - \hat{b} + w_i \\ &= \hat{z}_i(\hat{x}, \hat{b}) + d_x(\hat{x}, x_{S_i})'(x - \hat{x}) + b - \hat{b} + w_i \\ &= \hat{z}_i(\hat{x}, \hat{b}) + [d_x(\hat{x}, x_{S_i})' \;\; 1] \begin{bmatrix} x - \hat{x} \\ b - \hat{b} \end{bmatrix} + w_i \qquad i = 1, \ldots, 6 \end{aligned}$$

Following the j-th iteration of the ILS, the linearized system can be written as

$$\triangle z_i(\hat{x}_j) \triangleq z_i - \hat{z}_i(\hat{x}_j) = [d_x(\hat{x}_j, x_{S_i})' \;\; 1] \begin{bmatrix} x - \hat{x}_j \\ b - \hat{b}_j \end{bmatrix} + w_i \qquad i = 1, \ldots, 6$$

Using all the available measurements, one has equation:

$$\begin{aligned} \triangle z(\hat{x}_j) &= [\triangle z_1(\hat{x}_j) \ldots \triangle z_6(\hat{x}_j)]' \\ &= \begin{bmatrix} d_x(\hat{x}_j, x_{S_1})' & 1 \\ \vdots & \\ d_x(\hat{x}_j, x_{S_6})' & 1 \end{bmatrix} \begin{bmatrix} x - \hat{x}_j \\ b - \hat{b}_j \end{bmatrix} + w \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} d_x(\hat{x}_j, x_{S_1})' & 1 \\ \vdots & \\ d_x(\hat{x}_j, x_{S_6})' & 1 \end{bmatrix} (\mathbf{x} - \hat{\mathbf{x}}_j) + w \\
&\triangleq \mathbf{H}(\hat{x}_j)(\mathbf{x} - \hat{\mathbf{x}}_j) + w
\end{aligned}
$$

where w is the 6-dimensional stacked vector of the measurement noises and

$$
\mathbf{H}(\hat{x}_j) \triangleq \begin{bmatrix} d_x(\hat{x}_j, x_{S_1})' & 1 \\ \vdots & \\ d_x(\hat{x}_j, x_{S_6})' & 1 \end{bmatrix}
$$

is the augmented measurement matrix.

Thus, using the ILS method, the next estimate $\hat{\mathbf{x}}_{j+1}$ follows from

$$
\begin{aligned}
\hat{\mathbf{x}}_{j+1} - \hat{\mathbf{x}}_j &\triangleq \begin{bmatrix} \hat{x}_{j+1} - \hat{x}_j \\ \hat{b}_{j+1} - \hat{b}_j \end{bmatrix} \\
&= [\mathbf{H}(\hat{x}_j)' R^{-1} \mathbf{H}(\hat{x}_j)]^{-1} \mathbf{H}(\hat{x}_j)' R^{-1} \triangle z(\hat{x}_j)
\end{aligned}
$$

where

$$
R = E[ww'] = \sigma^2 I
$$

is the measurement noise covariance matrix.

The recursion for the ILS is then

$$
\hat{\mathbf{x}}_{j+1} = \hat{\mathbf{x}}_j + [\mathbf{H}(\hat{x}_j)' R^{-1} \mathbf{H}(\hat{x}_j)]^{-1} \mathbf{H}(\hat{x}_j)' R^{-1} \triangle z(\hat{x}_j)
$$

Since R is proportional to an identity matrix

$$
\hat{\mathbf{x}}_{j+1} = \hat{\mathbf{x}}_j + [\mathbf{H}(\hat{x}_j)' \mathbf{H}(\hat{x}_j)]^{-1} \mathbf{H}(\hat{x}_j)' \triangle z(\hat{x}_j)
$$

(ii) For this problem's geometry, the algorithm usually converges after the 3rd iteration. The MSE and RMSE values obltained from $N = 100$ runs are shown in Table 1 below.

(iii) The covariance of the estimate is

$$
P = [\mathbf{H}(\hat{x}_n)' R^{-1} \mathbf{H}(\hat{x}_n)]^{-1}
$$

where n is the last iteration. The diagonal terms of this matrix are shown in Table 1 under TMSE (theoretical MSE).

From (2.6.3-6), the sample MSE from N runs should be (with 95% probability) within

$$
2\sqrt{2P_{ii}^2/N} \approx 0.3 P_{ii}
$$

i.e., within 30% of the corresponding theoretical values P_{ii}, the diagonal terms of the calculated covariance of the estimate. The sample MSEs in Table 1 satisfy this and thus they are statistically compatible.

	MSE (m^2)	RMSE (m)	TMSE (m^2)
x	59	7.7	51
y	50	7.1	50
z	140	11.8	38
b	30	5.5	38

Table 1: Results for satellites 1–6.

(iv)

$$\mathrm{RMSE}_{\mathrm{horiz}} = \sqrt{P_{11} + P_{22}} = \sqrt{51 + 50} = 10\mathrm{m}$$

$$\mathrm{HDOP} = \frac{\sqrt{P_{11} + P_{22}}}{\sigma_w} = \frac{10}{10} = 1$$

$$\mathrm{VDOP} = \frac{\sqrt{P_{33}}}{\sigma_w} = \frac{\sqrt{38}}{10} = 0.6$$

(v) The results are shown in Table 2.

	MSE (m^2)	RMSE (m)	TMSE (m^2)
x	82	9	72
y	82	9	70
z	4866	70	5307
b	118	10.9	125

Table 2: Results for satellites 2–5.

The sample MSEs in Table 2 are within 30% of the corresponding theoretical values P_{ii}, the diagonal terms of the calculated covariance of the estimate, and thus they are statistically compatible.

$$\mathrm{RMSE}_{\mathrm{horiz}} = \sqrt{P_{11} + P_{22}} = 12\mathrm{m}$$

$$\mathrm{HDOP} = \frac{\sqrt{P_{11} + P_{22}}}{\sigma_w} = \frac{12}{10} = 1.2$$

$$\mathrm{VDOP} = \frac{\sqrt{P_{33}}}{\sigma_w} = \frac{\sqrt{5307}}{10} = 7.2$$

The increase in the VDOP is due to the lack of the high orbit satellite — the altitude has to be estimated from medium orbit satellites, which have a poor geometry for the altitude, yielding a (somewhat) ill-conditioned estimate of z.

ENGR352 Final Problem SOLUTION

Multisource Information Fusion

(i)

$$\lambda = 0.25 \cdot 4/10 = 0.1$$

For such a low value of λ, one can use (6.5.3-29)–(6.5.3-30)

$$\alpha \approx \sqrt{2\lambda}$$

$$\beta \approx \lambda$$

The s.s. covariance of the filtered state is

$$\bar{P} = \begin{bmatrix} \alpha\sigma_w^2 & \frac{\beta}{T}\sigma_w^2 \\ \frac{\beta}{T}\sigma_w^2 & \frac{\beta}{T^2}\frac{\alpha-\beta/2}{1-\alpha}\sigma_w^2 \end{bmatrix} = \begin{bmatrix} 31.62 & 5 \\ 5 & 0.97 \end{bmatrix}$$

(ii)

$$P_{xy} = \bar{P}C'$$

$$P_{yy} = C\bar{P}C' + \sigma_e^2$$

$$\hat{x} = \bar{x} + P_{xy}P_{yy}^{-1}(y - C\bar{x})$$

$$\hat{P} = \bar{P} - P_{xy}P_{yy}^{-1}P_{xy}'$$

(iii)

$$\hat{P} = \begin{bmatrix} 11.78 & 1.76 \\ 1.76 & 0.45 \end{bmatrix}$$

(iv)

$$\hat{P} = \begin{bmatrix} 16.73 & 2.75 \\ 2.75 & 0.63 \end{bmatrix}$$

(v) The first case. This is due to the fact that the two state components (position and velocity) have positively correlated errors in $\bar{P}$ while the measurement of their sum “hides” errors of opposite sign in these two variables (if one is larger and the other is smaller by the same quantity, this is not “seen” in such a measurement), i.e., the errors will have negative correlation. The positive correlation from the prior $\bar{P}$ and the negative correlation from the external measurement cancel (partially) each other, yielding smaller final errors.